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On The Algebraic Properties of Symbolic 6-Plithogenic Integers

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Abstract: This paper is dedicated to study the properties of symbolic 6-plithogenic integers and number theory, where we present many numbers theoretical concepts such as symbolic 6-plithogenic congruencies, symbolic 6-plithogenic Diophantine equations, and symbolic 6-plithogenic Euler's function with Euclidean division. Also, we present many examples to explain the validity and the scientific contribution of our work.

Keywords: symbolic 6-plithogenic integer, symbolic 6-plithogenic congruencies, symbolic 6-plithogenic division.

Introduction

Symbolic n-plithogenic sets were defined for the first time by Smarandache in [4, 24-25], with many interesting algebraic properties.

In [1-3], the symbolic 2-plithogenic rings were defined as an extension of classical rings. Many results were obtained with respect to their ideals and homomorphisms. The symbolic 2-plithogenic rings and fields have many applications in generalizing other algebraic structures such as symbolic 2-plithogenic vector spaces, symbolic 2-plithogenic modules, and symbolic 2-plithogenic equations [5-7].

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Laterally, many authors defined and studied symbolic 3-plithogenic algebraic structures, such as symbolic 3-plithogenic spaces and modules, see [8, 21-23].

In the literature, the extended integer systems were used in number theory, for example neutrosophic numbers have helped with neutrosophic number theory, refined neutrosophic numbers generated refined number theory and split-complex numbers generated split-complex number theory [9-20].

This has motivated many authors to study symbolic 2-plithogenic and symbolic 3-plithogenic number theoretical concepts such as congruencies, and Diophantine equations [26-36]. The generalized versions of number theoretical concepts are very applicable in other mathematical studies, especially in cryptography.

In this paper, we study the symbolic 6-plithogenic number theoretical concepts for the first time, and we illustrated many examples to clarify the novel approach.

Main discussion

Definition:

The rung of symbolic 6-plithogenic integer is defined as follows:

$6-SP_Z=\{x_0+\sum_{i=1}^{6}x_i\,P_i; x_i\in Z\}$, where $P_i\times P_j=p_{max(i,j)}, {P_i}^2=P_i$.

Definition.

Let $X=x_0+\sum_{i=1}^{6}x_i\,P_i, Y=y_0+\sum_{i=1}^{6}y_i\,P_i, Z=z_0+\sum_{i=1}^{6}z_i\,P_i\in 6-SP_Z$, we say that:

1). $X\setminus Y$ if there exists $Z\in 6-SP_Z$ such that $X.Z=Y$.

2). $X\equiv Y(mod\ Z)$ if $Z\setminus X-Y$.

3). $Z=gcd(X,Y)$ if $Z\setminus X, Z\setminus Y$ and if $T\setminus X, T\setminus Y$, then $T\setminus Z$.

4). X,Y are realtively prime if $gcd(X,Y)=1$.

Theorem1.

Let $X=x_0+\sum_{i=1}^{6}x_i\,P_i, Y=y_0+\sum_{i=1}^{6}y_i\,P_i, Z=z_0+\sum_{i=1}^{6}z_i\,P_i\in 6-SP_Z$, then:

1). $Z=gcd(X,Y)$ if and only if:

$$\begin{cases} z_0=gcd(x_0,y_0) \\ \sum_{i=0}^{j}z_i=gcd\left(\sum_{i=o}^{j}x_i,\sum_{i=0}^{j}y_i\right); 1\le j\le 6\end{cases}$$

2). $X \equiv Y(mod\, Z)$ if and only if $\sum_{i=0}^{j} x_i \equiv \sum_{i=0}^{j} y_i \left(mod \sum_{i=0}^{j} z_i\right)$, where $0 \leq j \leq 6$.

3). If $X \setminus Y$ then $\sum_{i=o}^{j} x_i \setminus \sum_{i=0}^{j} y_i\, ; 0 \leq j \leq 6$.

Theorem2.

Let $X = x_0 + \sum_{i=1}^{6} x_i P_i, Y = y_0 + \sum_{i=1}^{6} y_i P_i, Z = z_0 + \sum_{i=1}^{6} z_i P_i, A = a_0 + \sum_{i=1}^{6} a_i P_i, B = b_0 + \sum_{i=1}^{6} b_i P_i, C = c_0 + \sum_{i=1}^{6} c_i P_i \in 6 - SP_Z$, then:

1). If $Z \setminus X, Z \setminus Y$, then $Z \setminus AX + BY$.

2). If $Z = gcd(X, Y)$, then there exists $A, B \in 6 - SP_Z$ such that $AX + BY = Z$.

3). If $X \equiv Y(mod\, Z)$, then:

$$\begin{cases} X + C = Y + C\ (mod\, Z) \quad (I) \\ X - C = Y - C\ (mod\, Z) \quad (II) \\ X.C = Y.C\ (mod\, Z) \quad (III) \end{cases}$$

4). X is invertible modulo Z if and only if $\sum_{i=o}^{j} x_i$ is invertible modulo$\sum_{i=o}^{j} z_i\, ; 0 \leq j \leq 6$, and:

$X^{-1}(mod\, Z) = {x_0}^{-1}(mod\, z_0) + P_1[(x_0 + x_1)^{-1}(mod\, z_0 + z_1) - {x_0}^{-1}(mod\, z_0)] + P_2[(x_0 + x_1 + x_2)^{-1}(mod\, z_0 + z_1 + z_2) - (x_0 + x_1)^{-1}(mod\, z_0 + z_1)] + P_3[(x_0 + x_1 + x_2 + x_3)^{-1}(mod\, z_0 + z_1 + z_2 + z_3) - (x_0 + x_1 + x_2)^{-1}(mod\, z_0 + z_1 + z_2)] + P_4[(x_0 + x_1 + x_2 + x_3 + x_4)^{-1}(mod\, z_0 + z_1 + z_2 + z_3 + z_4) - (x_0 + x_1 + x_2 + x_3)^{-1}(mod\, z_0 + z_1 + z_2 + z_3)] + P_5[(x_0 + x_1 + x_2 + x_3 + x_4 + x_5)^{-1}(mod\, z_0 + z_1 + z_2 + z_3 + z_4 + z_5) - (x_0 + x_1 + x_2 + x_3 + x_4)^{-1}(mod\, z_0 + z_1 + z_2 + z_3 + z_4)] + P_6[(x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6)^{-1}(mod\, z_0 + z_1 + z_2 + z_3 + z_4 + z_5 + z_6) - (x_0 + x_1 + x_2 + x_3 + x_4 + x_5)^{-1}(mod\, z_0 + z_1 + z_2 + z_3 + z_4 + z_5)]$.

Theorem3.

Let $AX + BY = C$ be symbolic 6-plithogenic Diophantine equation in two variables, $A, B, C, X, Y \in 6 - SP_Z$, hence it is solvable if and only if:

$\sum_{i=0}^{j} a_i \sum_{i=0}^{j} x_i + \sum_{i=0}^{j} b_i \sum_{i=0}^{j} y_i = \sum_{i=0}^{j} c_i\, ; 0 \leq j \leq 6$ are solvable, i.e. $gcd\left(\sum_{i=0}^{j} a_i, \sum_{i=0}^{j} b_i\right) \setminus \sum_{i=0}^{j} c_i\, ; 0 \leq j \leq 6$.

Theorem4.

Let $X = x_0 + \sum_{i=1}^{6} x_i p_i \in 6 - SP_Z$, then:

$$X^n = {x_0}^n + P_1\left[\left(\sum_{i=0}^{1} x_i\right)^n - {x_0}^n\right] + P_2\left[\left(\sum_{i=0}^{2} x_i\right)^n - \left(\sum_{i=0}^{1} x_i\right)^n\right]$$
$$+ P_3\left[\left(\sum_{i=0}^{3} x_i\right)^n - \left(\sum_{i=0}^{2} x_i\right)^n\right] + P_4\left[\left(\sum_{i=0}^{4} x_i\right)^n - \left(\sum_{i=0}^{3} x_i\right)^n\right]$$
$$+ P_5\left[\left(\sum_{i=0}^{5} x_i\right)^n - \left(\sum_{i=0}^{4} x_i\right)^n\right] + P_6\left[\left(\sum_{i=0}^{6} x_i\right)^n - \left(\sum_{i=0}^{5} x_i\right)^n\right]$$

Theorem5.

(X, Y, Z) is a symbolic 6-plithogenic Pythagoras triple i.e. it is a solution of the non linear Diophantine equation $X^2 + Y^2 = Z^2$, if and only if $\left(\sum_{i=0}^{j} x_i, \sum_{i=0}^{j} y_i, \sum_{i=0}^{j} z_i\right); 0 \le j \le 6$ is a Pythagoras triple in Z.

Theorem6.

(X, Y, Z, T) is a symbolic 6-plithogenic Pythagoras quadruple i.e. it is a solution of the non linear Diophantine equation $X^2 + Y^2 + Z^2 = T^2$, if and only if $\left(\sum_{i=0}^{j} x_i, \sum_{i=0}^{j} y_i, \sum_{i=0}^{j} z_i, \sum_{i=0}^{j} t_i\right); 0 \le j \le 6$ is a Pythagoras quadruple in Z.

Proof of theorem1.

1). We put

$$Z = z_0 + \sum_{i=1}^{6} z_i P_i, z_0 = gcd(x_0, y_0), \sum_{i=1}^{1} z_i = gcd\left(\sum_{i=1}^{1} x_i, \sum_{i=1}^{1} y_i\right), \sum_{i=1}^{2} z_i$$
$$= gcd\left(\sum_{i=1}^{2} x_i, \sum_{i=1}^{2} y_i\right)$$
$$\sum_{i=1}^{3} z_i = gcd\left(\sum_{i=1}^{3} x_i, \sum_{i=1}^{3} y_i\right), \sum_{i=1}^{4} z_i = gcd\left(\sum_{i=1}^{4} x_i, \sum_{i=1}^{4} y_i\right), \sum_{i=1}^{5} z_i$$
$$= gcd\left(\sum_{i=1}^{5} x_i, \sum_{i=1}^{5} y_i\right), \sum_{i=1}^{6} z_i = gcd\left(\sum_{i=1}^{6} x_i, \sum_{i=1}^{6} y_i\right)$$

Assume that $T = t_0 + \sum_{i=1}^{6} t_i P_i$ with $T \setminus X, T \setminus Y$, hence:

$$\begin{cases} \sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} x_i , \sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} y_i ; 0 \le j \le 6 \\ \sum_{i=0}^{j} t_i \setminus \sum_{i=0}^{j} x_i , \sum_{i=0}^{j} t_i \setminus \sum_{i=0}^{j} y_i ; 0 \le j \le 6 \end{cases}$$

So that $\sum_{i=0}^{j} t_i \setminus \sum_{i=0}^{j} z_i ; 0 \le j \le 6$, hence $T \setminus Z$ and $Z = gcd(X, Y)$.

2). $X \equiv Y (mod\ Z)$ if and only if $Z \setminus X - Y$, which is equivalent to

$\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} (x_i - y_i) ; 0 \le j \le 6$, hence $\sum_{i=0}^{j} x_i \equiv \sum_{i=0}^{j} y_i \left(mod\ \sum_{i=0}^{j} z_i\right); 0 \le j \le 6$.

3). Assume that $X \setminus Y$, hence:

$$\begin{cases} x_0z_0 = y_0 \quad (1) \\ x_0z_1 + x_1z_0 + x_1z_1 = y_1 \quad (2) \\ x_0z_2 + x_1z_2 + x_2z_2 + x_2z_0 + x_2z_1 = y_2 \quad (3) \\ x_0z_3 + x_1z_3 + x_2z_3 + x_3z_3 + x_3z_0 + x_3z_1 + x_3z_2 = y_3 \quad (4) \\ x_0z_4 + x_1z_4 + x_2z_4 + x_3z_4 + x_4z_4 + x_4z_0 + x_4z_1 + x_4z_2 + x_4z_3 = y_4 \quad (5) \\ x_0z_5 + x_1z_5 + x_2z_5 + x_3z_5 + x_4z_5 + x_5z_5 + x_5z_0 + x_5z_1 + x_5z_2 + x_5z_3 + x_5z_4 = y_5 \quad (6) \\ x_0z_6 + x_1z_6 + x_2z_6 + x_3z_6 + x_4z_6 + x_5z_6 + x_6z_6 + x_6z_0 + x_6z_1 + x_6z_2 + x_6z_3 + x_6z_4 + x_6z_5 = y_6 \quad (7) \end{cases}$$

By adding $(1) + (2), (1) + (2) + (3), (1) + (2) + (3) + (4), (1) + (2) + (3) + (4) + (5), (1) + (2) + (3) + (4) + (5) + (6)$, $(1) + (2) + (3) + (4) + (5) + (6) + (7)$ we get:

$$\begin{cases} x_0z_0 = y_0 \\ \sum_{i=1}^{1} x_i \sum_{i=1}^{1} z_i = \sum_{i=1}^{1} y_i \\ \sum_{i=1}^{2} x_i \sum_{i=1}^{2} z_i = \sum_{i=1}^{2} y_i \\ \sum_{i=1}^{3} x_i \sum_{i=1}^{3} z_i = \sum_{i=1}^{3} y_i \\ \sum_{i=1}^{4} x_i \sum_{i=1}^{4} z_i = \sum_{i=1}^{4} y_i \\ \sum_{i=1}^{5} x_i \sum_{i=1}^{5} z_i = \sum_{i=1}^{5} y_i \\ \sum_{i=1}^{6} x_i \sum_{i=1}^{6} z_i = \sum_{i=1}^{6} y_i \end{cases}$$

Which means that $\sum_{i=0}^{j} x_i \setminus \sum_{i=0}^{j} y_i ; 0 \le j \le 6$

Proof of theorem 2.

1). Assume that $Z \setminus X, Z \setminus Y$, then we get:

$\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} x_i$, and $\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} y_i\,; 0 \le j \le 5$.

So that $\sum_{i=0}^{j} z_i \setminus \left(\sum_{i=0}^{j} a_i \sum_{i=0}^{j} x_i + \sum_{i=0}^{j} b_i \sum_{i=0}^{j} y_i\right)$ for $0 \le j \le 5$ and $Z \setminus AX + BY$.

2). Assume that $Z = gcd(X, Y)$, then $\sum_{i=0}^{j} z_i = gcd\left(\sum_{i=0}^{j} x_i, \sum_{i=0}^{j} y_i\right)$ for all $0 \le j \le 5$.

According to Bezout's theorem, we can write:

There exists $a_j, b_j \in Z$ such that $\sum_{i=0}^{j} z_i = a_j \sum_{i=0}^{j} x_i + b_j \sum_{i=0}^{j} y_i$

by putting

$A = a_0 + (a_1 - a_0)P_1 + (a_2 - a_1)P_2 + (a_3 - a_2)P_3 + (a_4 - a_3)P_4 + (a_5 - a_4)P_5$,

$B = b_0 + (b_1 - b_0)P_1 + (b_2 - b_1)P_2 + (b_3 - b_2)P_3 + (b_4 - b_3)P_4 + (b_5 - b_4)P_5$, we get:

$Z = AX + BY$.

3). Assume that $X \equiv Y (mod\ Z)$, then:

$\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} (x_i - y_i)$ for all $0 \le j \le 6$, hence:

$$\begin{cases} \sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} (x_i - c_i + c_i - y_i) \\ \sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} (x_i + c_i - c_i + y_i) \end{cases}$$

Hence $X \pm C = Y \pm C (mod\ Z)$, also:

$\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} (x_i - y_i) \sum_{i=0}^{j} c_i$ i.e. $\sum_{i=0}^{j} z_i \setminus \sum_{i=0}^{j} x_i \sum_{i=0}^{j} c_i - \sum_{i=0}^{j} y_i \sum_{i=0}^{j} c_i$

Hence $X.C \equiv Y.C (mod\ Z)$.

4). X is invertible modulo Z If and only if there exists $Y = y_0 + \sum_{i=1}^{j} y_i\, p_i \in 6 - SP_Z$ such that $X.Y \equiv 1 (mod\ Z)$.

This equivalent to:

$\sum_{i=0}^{j} x_i \, . \sum_{i=0}^{j} y_i \equiv 1 (mod\ Z)$ for $0 \le j \le 6$, hence:

$\sum_{i=0}^{j} x_i$ is invertible modulo $\sum_{i=0}^{j} z_i$ and:

$$X^{-1} = {x_0}^{-1}(mod\ z_0) + P_1\left[\left(\sum_{i=0}^{1} x_i\right)^{-1}\left(mod\sum_{i=0}^{1} z_i\right) - {x_0}^{-1}(mod\ z_0)\right]$$
$$+ P_2\left[\left(\sum_{i=0}^{2} x_i\right)^{-1}\left(mod\sum_{i=0}^{2} z_i\right) - \left(\sum_{i=0}^{1} x_i\right)^{-1}\left(mod\sum_{i=0}^{1} z_i\right)\right]$$
$$+ P_3\left[\left(\sum_{i=0}^{3} x_i\right)^{-1}\left(mod\sum_{i=0}^{3} z_i\right) - \left(\sum_{i=0}^{2} x_i\right)^{-1}\left(mod\sum_{i=0}^{2} z_i\right)\right]$$
$$+ P_4\left[\left(\sum_{i=0}^{4} x_i\right)^{-1}\left(mod\sum_{i=0}^{4} z_i\right) - \left(\sum_{i=0}^{3} x_i\right)^{-1}\left(mod\sum_{i=0}^{3} z_i\right)\right]$$
$$+ P_5\left[\left(\sum_{i=0}^{5} x_i\right)^{-1}\left(mod\sum_{i=0}^{5} z_i\right) - \left(\sum_{i=0}^{4} x_i\right)^{-1}\left(mod\sum_{i=0}^{4} z_i\right)\right]$$
$$+ P_6\left[\left(\sum_{i=0}^{6} x_i\right)^{-1}\left(mod\sum_{i=0}^{6} z_i\right) - \left(\sum_{i=0}^{5} x_i\right)^{-1}\left(mod\sum_{i=0}^{5} z_i\right)\right]$$

Proof of theorem3.

It is easy to check that $AX + BY = C$ is equivalent to:

$$\sum_{i=0}^{j} a_i \sum_{i=0}^{j} x_i + \sum_{i=0}^{j} b_i \sum_{i=0}^{j} y_i = \sum_{i=0}^{j} c_i\,; 0 \le j \le 6$$

The previous six Diophantine equations are solvable if and only if:

$$gcd\left(\sum_{i=0}^{j} a_i\,, \sum_{i=0}^{j} b_i\right) \backslash \sum_{i=0}^{j} c_i\,; 0 \le j \le 6$$

proof on theorem4.

For $n = 1$, it holds directly.

We assume that it I true for k, we prove it for $k + 1$.

$$X^{k+1} = XX^k = \left(x_0 + \sum_{i=0}^{6} x_i\, p_i\right)\left[x_0{}^k + P_1\left(\left(\sum_{i=0}^{1} x_i\right)^k - x_0{}^k\right)\right.$$
$$+ P_2\left(\left(\sum_{i=0}^{2} x_i\right)^k - \left(\sum_{i=0}^{1} x_i\right)^k\right) + P_3\left(\left(\sum_{i=0}^{3} x_i\right)^k - \left(\sum_{i=0}^{2} x_i\right)^k\right)$$
$$+ P_4\left(\left(\sum_{i=0}^{4} x_i\right)^k - \left(\sum_{i=0}^{3} x_i\right)^k\right) + P_5\left(\left(\sum_{i=0}^{5} x_i\right)^k - \left(\sum_{i=0}^{4} x_i\right)^k\right)$$
$$\left. + P_6\left(\left(\sum_{i=0}^{6} x_i\right)^k - \left(\sum_{i=0}^{5} x_i\right)^k\right)\right]$$
$$= x_0{}^{k+1} + P_1\left[x_0{}^k\left(\sum_{i=0}^{1} x_i\right)^k - x_0{}^{k+1} + x_1 x_0{}^k + x_1\left(\sum_{i=0}^{1} x_i\right)^k - x_1 x_0{}^k\right]$$
$$+ P_2\left[x_0\left(\sum_{i=0}^{2} x_i\right)^k - x_0\left(\sum_{i=0}^{1} x_i\right)^k + x_1\left(\sum_{i=0}^{2} x_i\right)^k - x_1\left(\sum_{i=0}^{1} x_i\right)^k + x_2 x_0{}^k\right.$$
$$\left. + x_1\left(\sum_{i=0}^{1} x_i\right)^k - x_2 x_0{}^k + x_2\left(\sum_{i=0}^{2} x_i\right)^k - x_2\left(\sum_{i=0}^{1} x_i\right)^k\right]$$
$$+ P_3\left[x_0\left(\sum_{i=0}^{3} x_i\right)^k - x_0\left(\sum_{i=0}^{2} x_i\right)^k + x_1\left(\sum_{i=0}^{3} x_i\right)^k - x_1\left(\sum_{i=0}^{2} x_i\right)^k\right.$$
$$+ x_2\left(\sum_{i=0}^{3} x_i\right)^k - x_2\left(\sum_{i=0}^{2} x_i\right)^k + x_2 x_0{}^k + x_3\left(\sum_{i=0}^{1} x_i\right)^k - x_3 x_0{}^k$$
$$\left. + x_3\left(\sum_{i=0}^{2} x_i\right)^k - x_2\left(\sum_{i=0}^{1} x_i\right)^k + x_3\left(\sum_{i=0}^{3} x_i\right)^k - x_2\left(\sum_{i=0}^{2} x_i\right)^k\right] + \cdots$$
$$= x_0{}^{k+1} + +P_1\left[\left(\sum_{i=0}^{1} x_i\right)^{k+1} - x_0{}^{k+1}\right] + P_2\left[\left(\sum_{i=0}^{2} x_i\right)^{k+1} - \left(\sum_{i=0}^{1} x_i\right)^{k+1}\right]$$
$$+ \cdots$$

And the proof holds.

Proof of theorem5.

$X^2 + Y^2 = Z^2$ implies that:

$$\begin{cases} {x_0}^2 + {y_0}^2 = {z_0}^2 \\ \left(\sum_{i=0}^{1} x_i\right)^2 + \left(\sum_{i=0}^{1} y_i\right)^2 = \left(\sum_{i=0}^{1} z_i\right)^2 \\ \left(\sum_{i=0}^{2} x_i\right)^2 + \left(\sum_{i=0}^{2} y_i\right)^2 = \left(\sum_{i=0}^{2} z_i\right)^2 \\ \left(\sum_{i=0}^{3} x_i\right)^2 + \left(\sum_{i=0}^{3} y_i\right)^2 = \left(\sum_{i=0}^{3} z_i\right)^2 \\ \left(\sum_{i=0}^{4} x_i\right)^2 + \left(\sum_{i=0}^{4} y_i\right)^2 = \left(\sum_{i=0}^{4} z_i\right)^2 \\ \left(\sum_{i=0}^{5} x_i\right)^2 + \left(\sum_{i=0}^{5} y_i\right)^2 = \left(\sum_{i=0}^{5} z_i\right)^2 \\ \left(\sum_{i=0}^{6} x_i\right)^2 + \left(\sum_{i=0}^{6} y_i\right)^2 = \left(\sum_{i=0}^{6} z_i\right)^2 \end{cases}$$

Which implies the proof.

Theorem 6 can be proved by the same argument.

Definition.

Let $X = x_0 + \sum_{i=0}^{6} x_i P_i \in 6 - SP_Z$, hence we say that $X > 0$ if and only if $x_0 > 0, \sum_{i=0}^{k} x_i > 0\ ; 1 \le k \le 6$

For example: $X = 3 + P_1 - P_2 + 2P_3 - P_4 - P_5 > 0$, that is because:

$3 > 0, 4 > 0, 3 > 0, 5 > 0, 4 > 0, 3 > 0$.

If $Y = y_0 + \sum_{i=0}^{6} y_i P_i \in 6 - SP_Z$, we ay that $X \ge Y$ if and only if $x_0 \ge y_0, \sum_{i=0}^{k} x_i \ge \sum_{i=0}^{k} y_i\ ; 1 \le k \le 6$.

For $X = 2 + P_1 + 2P_2 + 5P_3 + P_4 + 6P_5, Y = 1 + P_1 + P_2 + P_3 + 3P_4 + P_5, X \ge Y$, that is because:

$2 \ge 1, 3 \ge 2, 5 \ge 3, 10 \ge 4, 11 \ge 7, 17 \ge 8$

Definition.

Let $X = x_0 + \sum_{i=0}^{6} x_i P_i\ , y = y_0 + \sum_{i=0}^{6} y_i\, P_i \ge 0$, hence:

$$X^Y = {x_0}^{y_0} + P_1\left[\left(\sum_{i=0}^{1} x_i\right)^{\sum_{i=0}^{1} y_i} - {x_0}^{y_0}\right] + P_2\left[\left(\sum_{i=0}^{2} x_i\right)^{\sum_{i=0}^{2} y_i} - \left(\sum_{i=0}^{1} x_i\right)^{\sum_{i=0}^{1} y_i}\right]$$
$$+ P_3\left[\left(\sum_{i=0}^{3} x_i\right)^{\sum_{i=0}^{3} y_i} - \left(\sum_{i=0}^{2} x_i\right)^{\sum_{i=0}^{2} y_i}\right]$$
$$+ P_4\left[\left(\sum_{i=0}^{4} x_i\right)^{\sum_{i=0}^{4} y_i} - \left(\sum_{i=0}^{3} x_i\right)^{\sum_{i=0}^{3} y_i}\right]$$
$$+ P_5\left[\left(\sum_{i=0}^{5} x_i\right)^{\sum_{i=0}^{5} y_i} - \left(\sum_{i=0}^{4} x_i\right)^{\sum_{i=0}^{4} y_i}\right]$$
$$+ P_6\left[\left(\sum_{i=0}^{6} x_i\right)^{\sum_{i=0}^{6} y_i} - \left(\sum_{i=0}^{5} x_i\right)^{\sum_{i=0}^{5} y_i}\right]$$

Definition.

Let $X = x_0 + \sum_{i=0}^{6} x_i P_i > 0$, then:

$$\varphi(X) = \varphi(x_0) + P_1\left[\varphi\left(\sum_{i=0}^{1} x_i\right) - \varphi(x_0)\right] + P_2\left[\varphi\left(\sum_{i=0}^{2} x_i\right) - \varphi\left(\sum_{i=0}^{1} x_i\right)\right]$$
$$+ P_3\left[\varphi\left(\sum_{i=0}^{3} x_i\right) - \varphi\left(\sum_{i=0}^{2} x_i\right)\right] + P_4\left[\varphi\left(\sum_{i=0}^{4} x_i\right) - \varphi\left(\sum_{i=0}^{3} x_i\right)\right]$$
$$+ P_5\left[\varphi\left(\sum_{i=0}^{5} x_i\right) - \varphi\left(\sum_{i=0}^{4} x_i\right)\right] + P_6\left[\varphi\left(\sum_{i=0}^{6} x_i\right) - \varphi\left(\sum_{i=0}^{5} x_i\right)\right]$$

Where φ is Euler's function on Z.

Theorem.

Let $X = x_0 + \sum_{i=0}^{6} x_i {P_i}_i, Y = y_0 + \sum_{i=0}^{6} y_i P_i \in 6 - SP_Z, gcd(X,Y) = 1$ and $X, Y > 0$, hence:

$X^{\varphi(Y)} \equiv 1(mod\ Y)$

Proof.

$gcd(x_0, y_0) = 1$, hence ${x_0}^{\varphi(y_0)} \equiv 1(mod\ y_0)$.

$gcd(\sum_{i=0}^{1} x_i, \sum_{i=0}^{1} y_i) = 1$, hence $(\sum_{i=0}^{1} x_i)^{\varphi(\sum_{i=0}^{1} y_i)} \equiv 1(mod\ \sum_{i=0}^{1} y_i)$

By a similar argument, we get:

$$\left(\sum_{i=0}^{2} x_i\right)^{\varphi\left(\sum_{i=0}^{2} y_i\right)} \equiv 1\left(mod \sum_{i=0}^{2} y_i\right), \left(\sum_{i=0}^{3} x_i\right)^{\varphi\left(\sum_{i=0}^{3} y_i\right)} \equiv 1\left(mod \sum_{i=0}^{3} y_i\right)$$

$$\left(\sum_{i=0}^{4} x_i\right)^{\varphi\left(\sum_{i=0}^{4} y_i\right)} \equiv 1\left(mod \sum_{i=0}^{4} y_i\right), \left(\sum_{i=0}^{5} x_i\right)^{\varphi\left(\sum_{i=0}^{5} y_i\right)} \equiv 1\left(mod \sum_{i=0}^{5} y_i\right), \left(\sum_{i=0}^{6} x_i\right)^{\varphi\left(\sum_{i=0}^{6} y_i\right)} \equiv 1\left(mod \sum_{i=0}^{6} y_i\right)$$

This implies

$X^{\varphi(Y)} \equiv 1 + (1-1)P_1 + (1-1)P_2 + (1-1)P_3 + (1-1)P_4 + (1-1)P_5 +$
$(1-1)P_6 \equiv 1(mod\ Y)$.

Remark.

We call previous result by symbolic 6-plithogenic Euler's theorem.

Conclusion

In this work, we have studied the properties of symbolic 6-plithogenic integers for the first time, where concepts such as symbolic 6-plithogenic divisors, congruencies, and linear Diophantine equations were handled by many theorems and examples.

Also, we have presented the conditions of symbolic 6-plithogenic Pythagoras triples and quadruples in the corresponding symbolic 6-plithogenic ring of integers.

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